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Abstract

We discuss a general formalism needed for a unified description of the weak and electromagnetic processes in nuclei which is based on the multipole decomposition of the hadronic currents. The use of harmonic oscillator single-particle basis, which is commonly employed in many-body nuclear calculations, simplifies the relevant expressions. We present analytic formulas for the corresponding radial integrals which enter the transition matrix elements of one-body operators in various semi-leptonic nuclear processes. As an example, we apply our formalism in order to simplify previous formalism giving the nuclear moments for the $0\nu\beta\beta$ - decay. Our results for the radial integrals refer to the 1s-0d-1p-0f model space.

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1 Introduction

In a unified treatment of the semi-leptonic electroweak interactions in nuclei, like lepton-nucleus scattering [1, 2], β-decay modes [3], charged-lepton nuclear capture (μ^- capture, etc.) [4], neutrino-nucleus induced reactions [2, 5, 6], exotic semi-leptonic processes in nuclei (double beta-decay [7, 8, 9], $\mu^- \to e^-$ conversion, etc. [10, 11]), at first, one takes advantage of the well known electromagnetic interactions to probe the nucleus and determine accurately the nuclear charge, convection current and spin magnetization distributions [1, 2, 6, 12]. Afterwards, having the nuclear structure uncertainties reduced to a minimum, one can predict the cross sections for a variety of semi-leptonic weak interaction processes.

As is well known, the reaction rates $\Gamma_{i\to f}$ between the initial $|i\rangle$ and a final $|f\rangle$ discrete nuclear states of any of the aforementioned processes is written in terms of the matrix elements of a specific effective Hamiltonians H_{eff} as

$$\Gamma_{i\to f} \sim |\langle f|H_{eff}|i\rangle|^2$$

where the H_{eff} can be assumed to be constructed within the current-current interaction hypothesis by first writing down the leptonic (j_λ) and hadronic (J_λ) currents.

The effective hadronic current-density operator $\hat{\mathcal{J}}_\lambda$ which enters the H_{eff}, contains only strong isoscalar ($T = 0$) and isovector ($T = 1$) pieces and, in general, it has both vector (V) and axial vector (A) pieces as

$$[\hat{\mathcal{J}}_\lambda]^{TM_T} = \beta_V^{(T)}[\hat{J}_\lambda]^{TM_T} + \beta_A^{(T)}[\hat{J}^5_\lambda]^{TM_T}. \qquad (1)$$

where $\beta_V^{(T)}$ and $\beta_A^{(T)}$ are polar-vector and axial-vector coefficients involving the relevant couplings (the purely electromagnetic interactions involve only vector current density, but the weak interactions involve in addition axial vector current). In neutral-current processes $M_T = 0$ for both isoscalar and isovector parts, while in charge changing interactions $T = 1$ with $M_T = \pm 1$. One, usually, assumes that for a nuclear target (denoted as (A, Z) with Z, A the proton- and mass-number respectively) the states $|i\rangle$ and $|f\rangle$ are characterized by definite angular momentum and parity ($|\alpha\rangle \equiv |J^\pi\rangle$). In neutral-current reactions (lepton scattering, $\mu - e$ conversion, etc.) the final state $|f\rangle$ is, in general, an excited state of the nucleus (A, Z) while in charged-current processes $|f\rangle$ denotes an excited state of the $(A, Z \mp 1)$ nucleus.

The nuclear calculations of $\langle f|H_{eff}|i\rangle$ proceed usually by decomposing the Fourier transform of the hadronic current-density operator into irreducible tensors in strong isospin space [2, 3]. Then, spherical Bessel functions $j_l(x)$ in central or non-central interaction components are obtained. The nuclear transition rates are subsequently determined by the matrix elements of these operators. The possibility to obtain closed expressions for the matrix elements of the basic one-body operators describing the above mentioned reactions has been examined in previous theoretical studies with special emphasis on the use of harmonic oscillator wave functions [7, 8, 9, 11, 12, 13].

In Refs. [10]-[13] a method was developed which provides explicit compact expressions for some basic nuclear form factors (those needed for the $\mu^- \to e^-$ conversion in nuclei, scattering of cold dark matter candidates off nuclei, etc.). The advantage of this method is that it allows the easy calculation of the required reduced matrix elements by separating the geometrical coefficients from the kinematical parameters of the reaction. In this way, the transition matrix elements for every value of the momentum transfer q can be evaluated immediately. It is worth remarking that such a compact formalism provides a very useful insight to those authors who wish to adopt a phenomenological approach and fit the nuclear transition strengths.

In the present work, we extend the method of Ref. [10]-[13] so as to provide expressions for the reduced matrix elements of any of the seven basic single-particle tensor operators describing the above-mentioned semi-leptonic processes [2, 5, 6]. We especially deal with the operators involving the differential operator ∇ (nabla), which have not been studied previously [11, 12, 13]. To this end, we first treat the corresponding radial matrix elements which contain derivatives with respect to the coordinate r. Finally, we discuss some representative examples as applications of our formalism focusing on the improvement of the previous expressions giving the radial part of the nuclear moments in the neutrinoless double-beta decay [7]-[9].

2 Multipole decomposition of the hadronic currents

From a nuclear physics point of view, the hadronic current (J_μ) which enters the weak and electromagnetic interaction Hamiltonian H_{eff} is of primary interest. In our convention the four-current operator $\hat{J}_\mu$ is written as

$$\hat{J}_\mu(\mathbf{r}) = (\hat{\rho}(\mathbf{r}), \hat{\mathbf{J}}(\mathbf{r})) \tag{2}$$

where $\hat{\rho}(\mathbf{r})$ the density and $\hat{\mathbf{J}}(\mathbf{r})$ the three-current operators. The standard multipole expansion procedure (see Appendix A) [2, 6], applied on the matrix elements of $\hat{J}_\mu(\mathbf{r})$, leads to spherical tensor operators which are given in terms of projection functions involving spherical Bessel functions $j_L(r)$ and spherical Harmonics $Y^L_M(\hat{r})$ or vector spherical Harmonics $\mathbf{Y}^{(L,1)J}_M(\hat{r})$ as:

$$M^J_M(q\mathbf{r}) \;=\; \delta_{LJ} j_L(qr) Y^L_M(\hat{r}), \tag{3}$$

$$\mathbf{M}^{(L1)J}_M(q\mathbf{r}) \;=\; j_L(qr)\mathbf{Y}^{(L1)J}_M(\hat{r}), \tag{4}$$

where

$$\mathbf{Y}^{(L1)J}_M(\hat{r}) \;=\; \sum_{m,q} \langle Lm1q|JM\rangle Y^L_m(\hat{r})\hat{e}_q, \tag{5}$$

$\hat{e}_q$ is a unit vector in the direction of the three-momentum transfer $\mathbf{q}$. The magnitude of $\mathbf{q}$ ($q = |\mathbf{q}|$) is given from the kinematics of the studied process (see e.g. Ref. [10]).

Using the projections functions Eqs. (3) and (4), in the case of the polar-vector current ($\hat{J}_\lambda$) we obtain the multipole operators

$$\hat{M}^{coul}_{JM;TM_T} \;=\; \int d\mathbf{r} M^J_M(q\mathbf{r})\hat{\rho}(\mathbf{r})_{TM_T}, \tag{6}$$

$$\hat{L}_{JM;TM_T} \;=\; i\int d\mathbf{r} \left(\frac{1}{q}\nabla M^J_M(q\mathbf{r})\right)\cdot \hat{\mathbf{J}}(\mathbf{r})_{TM_T}, \tag{7}$$

$$\hat{T}^{el}_{JM;TM_T} \;=\; \int d\mathbf{r} \left(\frac{1}{q}\nabla \times \mathbf{M}^{JJ}_M(q\mathbf{r})\right)\cdot \hat{\mathbf{J}}(\mathbf{r})_{TM_T}, \tag{8}$$

$$\hat{T}^{mag}_{JM;TM_T} \;=\; \int d\mathbf{r}\mathbf{M}^{JJ}_M(q\mathbf{r})\cdot \hat{\mathbf{J}}(\mathbf{r})_{TM_T}. \tag{9}$$

which have well-defined parity and are called as: Coulomb, longitudinal, transverse-electric, and transverse-magnetic multipoles, respectively. The first three operators have parity $(-)^J$ (normal parity operators), while the parity of $\hat{T}^{mag}_J$ is $(-)^{J+1}$ (abnormal parity operator). The analogous axial-vector multipoles are

$$\hat{M}^5_{JM;TM_T} \;=\; \int d\mathbf{r} M^J_M(q\mathbf{r})\hat{\rho}(\mathbf{r})^5_{TM_T}, \tag{10}$$

$$\hat{L}^5_{JM;TM_T} \;=\; i\int d\mathbf{r} \left(\frac{1}{q}\nabla M^J_M(q\mathbf{r})\right)\cdot \hat{\mathbf{J}}(\mathbf{r})^5_{TM_T}, \tag{11}$$

$$\hat{T}^{el5}_{JM;TM_T} \;=\; \int d\mathbf{r} \left(\frac{1}{q}\nabla \times \mathbf{M}^{JJ}_M(q\mathbf{r})\right)\cdot \hat{\mathbf{J}}(\mathbf{r})^5_{TM_T}, \tag{12}$$

$$\hat{T}^{mag5}_{JM;TM_T} = \int d\mathbf{r}\mathbf{M}^{JJ}_M(q\mathbf{r}) \cdot \hat{\mathbf{J}}(\mathbf{r})^5_{TM_T}. \tag{13}$$

The first three axial-vector multipoles have parity $(-)^{J+1}$ while $\hat{T}^{mag5}_J$ is a normal parity operator. For a conserved vector current (CVC) like the electromagnetic, the longitudinal multipoles can be written in terms of the Coulomb ones as

$$\hat{L}_{JM_J;Tm_T}(q) = \frac{q_0}{q}\hat{M}_{JM_J;TM_T}(q) \tag{14}$$

(q_0 is the time component of the four-momentum transfer, $q_\mu = (q_0, \mathbf{q})$). In this case, the number of independent operators resulting from the projection procedure is reduced to seven.

The isospin dependence of the operators (6)-(13) includes the operator $I^{M_T}_T$ given by [2]

$$I^{M_T}_T = \begin{cases} 1, & T=0,\ M_T=0 \\ \tau_0 = \tau_3, & T=1,\ M_T=0 \\ \tau_\pm = \mp\frac{1}{\sqrt{2}}(\tau_1 \pm \tau_2), & T=1,\ M_T=\pm 1 \end{cases} \tag{15}$$

The exact form of $I^{M_T}_T$ is determined by the specific semi-leptonic reaction in question (see the Introduction).

The matrix elements of the seven basic operators of Eqs. (6)-(13) involve isospin dependent form factors $F^{(T)}_X$ (see Appendix A). For this reason, we define seven new operators of which the spin-space parts are written (in the first quantization) as [2, 5, 6]:

$$T^{JM}_1 \equiv M^J_M(q\mathbf{r}) = \delta_{LJ}\, j_L(\rho)Y^L_M(\hat{r}), \tag{16}$$

$$T^{JM}_2 \equiv \Sigma^J_M(q\mathbf{r}) = \mathbf{M}^{JJ}_M \cdot \boldsymbol{\sigma}, \tag{17}$$

$$T^{JM}_3 \equiv \Sigma'^J_M(q\mathbf{r}) = -i\left\{\frac{1}{q}\nabla \times \mathbf{M}^{JJ}_M(q\mathbf{r})\right\} \cdot \boldsymbol{\sigma}, \tag{18}$$

$$T^{JM}_4 \equiv \Sigma''^J_M(q\mathbf{r}) = \Big\{\frac{1}{q}\nabla M^J_M(q\mathbf{r})\Big\} \cdot \boldsymbol{\sigma}, \tag{19}$$

$$T^{JM}_5 \equiv \Delta^J_M(q\mathbf{r}) = \mathbf{M}^{JJ}_M(q\mathbf{r}) \cdot \frac{1}{q}\nabla, \tag{20}$$

$$T^{JM}_6 \equiv \Delta'^J_M(q\mathbf{r}) = -i\Big\{\frac{1}{q}\nabla \times \mathbf{M}^{JJ}_M(q\mathbf{r})\Big\} \cdot \nabla, \tag{21}$$

$$T^{JM}_7 \equiv \Omega^J_M(q\mathbf{r}) = M^J_M(q\mathbf{r})\boldsymbol{\sigma} \cdot \frac{1}{q}\nabla. \tag{22}$$

(hereafter we use the shorter notation for the projection functions of Eqs. (4) and (5) as $\mathbf{M}^{LJ}_M$ and $\mathbf{Y}^{LJ}_M$, respectively). For the reader's convenience, in Eqs. (16)-(22) we adopt the usual notation [2]. Using properties of the nabla operator (∇), Eqs. (18), (19) and (21) can be rewritten as:

$$T^{JM}_3 \equiv \Sigma'^J_M = [J]^{-1}\Big\{-J^{1/2}\mathbf{M}^{J+1J}_M + (J+1)^{1/2}\mathbf{M}^{J-1J}_M\Big\} \cdot \boldsymbol{\sigma}, \tag{23}$$

$$T_4^{JM} \equiv \Sigma''^J_M = [J]^{-1}\Big\{(J+1)^{1/2}\mathbf{M}_M^{J+1J} + J^{1/2}\mathbf{M}_M^{J-1J}\Big\}\cdot\boldsymbol{\sigma}, \tag{24}$$

$$T_6^{JM} \equiv \Delta'^J_M = [J]^{-1}\Big\{-J^{1/2}\mathbf{M}_M^{J+1J} + (J+1)^{1/2}\mathbf{M}_M^{J-1J}\Big\}\cdot\frac{1}{q}\nabla, \tag{25}$$

(throughout this work we use the common symbol $[J] = (2J+1)^{1/2}$). By glancing at Eqs. (16)-(25), we conclude that, the seven basic single-particle operators of definite parity are built with the aid of four operators

$$\mathcal{O}_1^{JM} = M_M^J, \quad \mathcal{O}_2^{JM} = \mathbf{M}_M^{LJ}\cdot\boldsymbol{\sigma}, \tag{26}$$

$$\mathcal{O}_3^{JM} = \mathbf{M}_M^{LJ}\cdot\frac{1}{q}\nabla, \quad \mathcal{O}_4^{JM} = M_M^J\boldsymbol{\sigma}\cdot\frac{1}{q}\nabla. \tag{27}$$

which contain the projection functions M_M^J and $\mathbf{M}_M^{LJ}$ or their products with the nucleon-spin $\boldsymbol{\sigma}$ and/or the ∇ operator.

3 The Single-particle matrix elements

Many quantities of the semi-leptonic electroweak processes are expressed (to a good approximation) in terms of single-particle nuclear matrix elements of the one-body operators $T_i^{JM}, i = 1, 2, ...7$ [2, 3, 5, 6]. These matrix elements, by applying the Wigner-Eckart theorem, are written as:

$$\langle j_1 m_1|T_i^{JM}|j_2 m_2\rangle = (-)^{j_1-m_1}\begin{pmatrix} j_1 & J & j_2 \\ -m_1 & M & m_2\end{pmatrix}\langle j_1||T_i^J||j_2\rangle, \tag{28}$$

In our notation, the single-particle wave functions are labelled (see Appendix B) with the quantum numbers $(nlsjm_j)$. The reduced matrix elements of the operators $\mathcal{O}_i^J$, $i = 1, 2, 3, 4$, after applying the re-coupling relations, can be written in the closed forms shown below.

1. For the operators $\mathcal{O}_i^J$, with $i = 1, 2$, the reduced matrix elements $\langle j_1||\mathcal{O}_i^{(L,S_i)J}||j_2\rangle$ have been compactly written as [11, 13]

$$\langle j_1||\mathcal{O}_i^{(L,S_i)J}||j_2\rangle = (l_1\ L\ l_2)\,\mathcal{U}_{LS}^J\langle n_1 l_1|j_L(\rho)|n_2 l_2\rangle, \quad i = 1, 2 \tag{29}$$

where the symbols $(l_1\ L\ l_2)$ and $\mathcal{U}_{LS}^J$ are

$$(l_1\ L\ l_2) \equiv (-)^{l_1}\frac{1}{\sqrt{4\pi}}[l_1][L][l_2]\begin{pmatrix} l_1 & L & l_2 \\ 0 & 0 & 0\end{pmatrix}, \tag{30}$$

$$\mathcal{U}_{LS}^J \equiv [j_1][j_2][J](S+1)^{1/2}(S+2)^{1/2}\begin{Bmatrix} l_1 & l_2 & L \\ 1/2 & 1/2 & S \\ j_1 & j_2 & J\end{Bmatrix}. \tag{31}$$

In the case of the Fermi-type operator $\mathcal{O}_1^J$, we must put $S = 0$ in the 9-j symbol of Eq. (31) while in the case of the Gamow-Teller operator $\mathcal{O}_2^J$, $S = 1$.

2. The reduced matrix elements of $\mathcal{O}_3^J$ after some manipulation can be cast in the form

$$\langle j_1||\mathbf{M}^{LJ}(q\mathbf{r})\cdot\frac{1}{q}\nabla||j_2\rangle = \sum_\alpha \mathcal{A}_L^\alpha(j_1 j_2; J)\langle n_1 l_1|\theta_L^\alpha(\rho)|n_2 l_2\rangle, \quad \alpha = \pm \tag{32}$$

where the coefficients $\mathcal{A}_L^\pm$ are

$$\begin{aligned}\mathcal{A}_L^\pm(j_1 j_2; J) &= \pm(-)^{l_1+L+j_2+1/2}[j_1][j_2][J]\left(\frac{2l_2+1\mp 1}{2}\right)^{1/2}(l_1\ L\ l_2\mp 1)\\ &\times\mathcal{W}_6(l_1, j_1, 1/2, j_2, l_2, J)\mathcal{W}_6(L, 1, J, l_2, l_1, l_2\mp 1),\end{aligned} \tag{33}$$

with $\mathcal{W}_6$ representing the common 6-j symbol

$$\mathcal{W}_6(l_1, j_1, 1/2, j_2, l_2, J) \equiv \begin{Bmatrix} l_1 & j_1 & 1/2 \\ j_2 & l_2 & J \end{Bmatrix}. \tag{34}$$

3. Similarly, for the reduced matrix element of $\mathcal{O}_4^J$ we can write

$$\langle j_1 || M^J(q\mathbf{r})\boldsymbol{\sigma} \cdot \frac{1}{q}\nabla || j_2 \rangle = \sum_\alpha \mathcal{B}_L^\alpha(j_1 j_2; J) \langle n_1 l_1 | \theta_J^\alpha(\rho) | n_2 l_2 \rangle, \quad \alpha = \pm \tag{35}$$

where

$$\mathcal{B}_L^\pm(j_1 j_2; J) = \pm\, \delta_{j_2, l_2 \mp 1/2}\, [j_1][j_2](l_1\ J\ 2j_2 - l_2)\, \mathcal{W}_6(l_1, j_1, 1/2, j_2, 2j_2 - l_2, J). \tag{36}$$

From Eqs. (29), (32) and (35) we notice that all basic single-particle reduced matrix elements required for our purposes rely on the following three types of radial integrals:

$$\langle n_1 l_1 | \theta_l^\alpha(\rho) | n_2 l_2 \rangle \equiv \int dr r^2 R^*_{n_1 l_1}(r) \theta_l^\alpha(\rho) R_{n_2 l_2}(r), \qquad \alpha = 0, \pm \tag{37}$$

with

$$\theta_l^0(\rho) = j_l(\rho), \qquad \theta_l^\pm(\rho) = j_l(\rho) \left(\frac{d}{d\rho} \pm \frac{2l_2 + 1 \pm 1}{2\rho} \right). \tag{38}$$

The argument ρ in Eqs. (38) is equal to $\rho = qr$. We note that the derivatives with respect to ρ appeared in the radial matrix elements come from the application of the gradient formula in the matrix elements of Eqs. (32) and (35).

3.1 Expressions for the Radial Integrals

It is well known that, for single-particle wave functions with arbitrary radial dependence it is not easy to perform analytically the integrations over r in Eq. (37). These integrals, however, can be simplified in the case when harmonic oscillator basis (see Appendix B) is used [2, 13]. Then they take the elegant expressions shown below.

(i) For the operator $\theta_l^0(\rho)$ in Ref. [13] it was proved that

$$\langle n_1 l_1 | j_L(\rho) | n_2 l_2 \rangle = e^{-y} y^{L/2} \sum_{\mu=0}^{n_{max}} \varepsilon_\mu^L\, y^\mu, \qquad y = (qb/2)^2 \tag{39}$$

$$n_{max} = (N_1 + N_2 - L)/2,$$

where $N_i = 2n_i + l_i$ represent the harmonic oscillator quanta of the i_{th} level. The coefficients $\varepsilon_\mu^L(n_1 l_1 n_2 l_2)$ are given by

$$\varepsilon_\mu^L(n_1 l_1 n_2 l_2) = G \frac{\pi^{1/2}}{2} \sum_{m_1=\phi}^{n_1} \sum_{m_2=\sigma}^{n_2} n! \Lambda_{m_1}(n_1 l_1) \Lambda_{m_2}(n_2 l_2) \Lambda_\mu(nL), \tag{40}$$

with

$$n = m_1 + m_2 + (l_1 + l_2 - L)/2.$$

Also, $G = b^3 N_{n_1 l_1} N_{n_2 l_2}/2$, where $N_{n_1 l_1}$ is defined in Appendix B. The other symbols of Eq. (40) are explained in Ref. [13].

(ii) The formulation of the radial matrix elements which include differential operators $\theta_l^{\pm}$, which constitute the main task of our present paper, proceeds in a similar manner to that of Eq. (39). This leads to the expressions

$$\langle n_1 l_1 | j_L(\rho) \left(\frac{d}{d\rho} \pm \frac{2l_2+1\pm 1}{2\rho} \right) | n_2 l_2 \rangle \;=\; e^{-y} y^{(L-1)/2} \sum_{\mu=0}^{n_{max}} \zeta_\mu^{\pm}(L)\, y^{\mu}, \tag{41}$$

where the geometrical coefficients $\zeta_\mu^{\pm}(n_1 l_1 n_2 l_2; L)$ are given in terms of those of Eq. (40), i.e.

$$\zeta_\mu^{-}(L) \;=\; -\frac{1}{2} \begin{cases} (n_2+l_2+3/2)^{1/2}\varepsilon_\mu^L(n_1 l_1 n_2 l_2+1) + n_2^{1/2}\varepsilon_\mu^L(n_1 l_1 n_2-1 l_2+1), & 0 \le \mu < n_{max} \\ (n_2+l_2+3/2)^{1/2}\varepsilon_{n_{max}}^L(n_1 l_1 n_2 l_2+1), & \mu = n_{max} \end{cases} \tag{42}$$

$$\zeta_\mu^{+}(L) \;=\; \frac{1}{2} \begin{cases} (n_2+l_2+1/2)^{1/2}\varepsilon_\mu^L(n_1 l_1 n_2 l_2-1) + (n_2+1)^{1/2}\varepsilon_\mu^L(n_1 l_1 n_2+1 l_2-1), & 0 \le \mu < n_{max} \\ (n_2+1)^{1/2}\varepsilon_{n_{max}}^L(n_1 l_1 n_2+1 l_2-1), & \mu = n_{max} \end{cases} \tag{43}$$

The value of the index n_{max} in the latter two cases of radial integrals is determined by

$$n_{max} = (N_1 + N_2 - L + 1)/2.$$

At this point, it should be noted that, the explicit and general formulas of Eqs. (40), (42) and (43) hold for every combination of the levels $(n_1 l_1)j_1$, $(n_2 l_2)j_2$, and give the geometrical (momentum independent) coefficients of the polynomials defined in Eqs. (39) and (41). These closed expressions have been constructed by inverting properly the multiple summations involved in the corresponding matrix elements (see Ref. [14, 15]), so as the final summation is performed over the harmonic oscillator quantum number $N = 2n + l$.

In order to show the advantages of the above formalism we discuss below some applications.

4 Applications

As mentioned before, our formalism is applicable for an infinite single-particle basis and can be applied to any semi-leptonic electroweak process which take place in the field of nuclei [14, 15]. In a realistic case one is forced to truncate the basis set. Thus, e.g. the description of the nuclear moments for the neutrinoless double beta decay $^{48}Ca \rightarrow ^{48}$Ti can be done using a model space including the seven orbitals of the major shells $2\hbar\omega$ and $3\hbar\omega$ [8, 9].

In Tables 1-3 we list the coefficients ε_μ^L and $\zeta_\mu^{\pm}$ needed to evaluate the radial matrix elements $\langle n_1 l_1 | \theta_L^{\alpha}(\rho) | n_2 l_2 \rangle$ in the above (1s-0d-1p-0f) model space. For some additional geometrical coefficients ε_μ^L, which determine the transition matrix elements for scattering of dark matter candidates off nuclei in the model space including the orbitals 2p-0h and 0i-1g, the reader is referred to Ref. [16].

By exploiting the analytic expressions of the Secs. 2 and 3, in the next subsection we simplify the formalism of Ref. [8, 9] constructed for the calculation of the nuclear moments in neutrinoless double-beta decay.

4.1 Nuclear moments for the neutrinoless double-beta decay

In the description of the nuclear moments for the neutrinoless double-beta decay [7]-[8] the following radial integrals are needed:

$$\mathcal{R}_L^{\kappa}(n_1 l_1 n_2 l_2; q) \;=\; \langle n_1 l_1 | j_L(qr) r^{\kappa} | n_2 l_2 \rangle \tag{44}$$

$$\mathcal{R}^{\kappa}_{L_1L_2}(n_1l_1n_2l_2, n'_1l'_1n'_2l'_2;\omega) \;=\; \int q^{2+\kappa}\upsilon(q;\omega)\mathcal{R}^{0}_{L_1}(n_1l_1n_2l_2;q)\mathcal{R}^{0}_{L_2}(n'_1l'_1n'_2l'_2;q)dq \qquad (45)$$

where $\upsilon(q;\omega)$ are functions describing the energy of the intermediate neutrinos in various gauge models (light-neutrino mixing, etc.) [3, 7, 8, 9]. The parameter ω is related to the excitation energy of the intermediate nucleus. Using the compact formalism presented in the previous section the above integrals are simplified and written as follows:

a. For the radial moments $\mathcal{R}^{\kappa}_{L}(n_1l_1n_2l_2;q)$ working as in Eq. (39) we find

$$\mathcal{R}^{\kappa}_{L}(n_1l_1n_2l_2;q) \;=\; b^{\kappa}y^{L/2}e^{-y}\sum_{\mu=0}^{n_{max}}\varepsilon^{L}_{\mu}y^{\mu} \qquad (46)$$

which implies that the nuclear moments $\mathcal{R}^{\kappa}_{L}(n_1l_1n_2l_2;q)$ are simply obtained with the aid of the coefficients $\varepsilon^{L}_{\mu}(n_1l_1n_2l_2)$ of Eq. (40). Note that for $\kappa = 0$ Eq. (46) reduces to Eq. (39).

b. By replacing the radial integrals in the right-hand side of Eq. (45) with Eq. (39) and manipulating their product we take

$$\mathcal{R}^{0}_{L_1}(n_1l_1n_2l_2;q)\mathcal{R}^{0}_{L_2}(n'_1l'_1n'_2l'_2;q) \;=\; e^{-2y}y^{(L_1+L_2)/2}\sum_{\mu=0}^{n_{max}}c_{\mu}y^{\mu} \qquad (47)$$

where

$$n_{max} = n_{1max} + n_{2max},$$

with

$$n_{1max} = n_1 + n_2 + (l_1 + l_2 - L_1)/2\,, \qquad n_{2max} = n'_1 + n'_2 + (l'_1 + l'_2 - L_2)/2$$

The coefficients c_{μ} in Eq. (47) are obtained from those of ε^{L}_{ν} as

$$c_{\mu} \;=\; \sum_{\lambda=0}^{\mu}\varepsilon^{L_1}_{\nu}\varepsilon^{L_2}_{\mu-\nu} \qquad (48)$$

In the last summation $\varepsilon^{L_1}_{\nu} \;=\; 0$, for $n_{1max} < \nu \leq n_{max}$, and $\varepsilon^{L_2}_{\mu-\nu} \;=\; 0$, for $n_{2max} < \mu - \nu \leq n_{max}$. By inserting Eq. (47) into Eq. (45) and putting $q \;=\; 2y^{1/2}/b$ [see Eq. (39)], we have

$$\mathcal{R}^{\kappa}_{L_1L_2}(n_1l_1n_2l_2, n'_1l'_1n'_2l'_2;\omega) \;=\; \frac{2^{\kappa+2}}{b^{\kappa+3}}\sum_{\mu=0}^{n_{max}}c_{\mu}\,\mathcal{I}^{\lambda}(\omega) \qquad (49)$$

where

$$\lambda = \mu + (L_1 + L_2 + \kappa + 1)/2$$

The quantities $\mathcal{I}^{\lambda}(\omega)$ represent the integrals

$$\mathcal{I}^{\lambda}(\omega) \;=\; \int_{0}^{\infty}\upsilon(y;\omega)e^{-2y}y^{\lambda}dy. \qquad (50)$$

which can be computed after providing the exact form of $\upsilon(y;\omega)$ [3]. Equation (49) is much simpler than Eq. (D.1) of Ref. [8], since the latter does not include a simple summation as Eq. (49), but a double one over similar integrals to those of Eq. (50).

5 Summary and Conclusions

Our main concern in the present paper was the general formalism needed for calculating the weak and electromagnetic processes in nuclei. Starting from the decomposition of the hadronic current into tensor multipole operators, we investigated the possibility of constructing analytic formulas for the matrix elements of the principal semi-leptonic operators. We payed special attention on the multipole matrix elements taken between single-particle states and restricted ourselves on the formulation of their radial part.

By utilizing harmonic oscillator basis, we have achieved all types of radial moments related to the seven basic one-body operators (involved in semi-leptonic reactions), to be written in the form of closed analytic expressions, i.e. as products of an exponential times a polynomial of even powers in the momentum transfer q. The coefficients of the polynomials found are in general simple numbers (for diagonal matrix elements they are always rational numbers) and they can be calculated with simple codes. We applied the above compact expressions in order to simplify the previous formalism [7, 8, 9] giving the nuclear moments in the interesting neutrinoless double-beta decay process. This helped us to see the advantages of the formalism described in Secs. 3 and 4.

6 Acknowledgements

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Appendix A

1. The identities which are needed in the multipole decomposition procedure are

$$exp(-i\mathbf{q}.\mathbf{r}) \;=\; 4\pi \sum_{JM_J} (-i)^J M_M^J(\mathbf{r}) Y_{JM}^*(\hat{e}_{q0}) \tag{51}$$

$$\hat{e}_{q\lambda} exp(-i\mathbf{q}.\mathbf{r}) \;\equiv\; -(2\pi)^{1/2} \sum_{J\geq 1} (-i)^J (2J+1)^{1/2} \Big[\lambda \mathbf{M}_M^{JJ}(\mathbf{r}) + \frac{1}{q}\nabla \times \mathbf{M}_M^{JJ}(\mathbf{r})\Big] \tag{52}$$

where $\lambda = \pm$ and the unit vector $\hat{e}_{q\lambda}$ has the spherical components

$$\hat{e}_{q0} \;\equiv\; \hat{e}_z \;=\; \mathbf{q}/q \tag{53}$$

$$\hat{e}_{q\pm} \;\equiv\; \mp \frac{1}{\sqrt{2}}(\hat{e}_x \pm i\hat{e}_y) \tag{54}$$

The components of the rank-one spherical tensor ∇_μ are

$$\nabla_0 \;=\; \frac{\partial}{\partial z} \tag{55}$$

$$\nabla_\pm \;=\; \mp \left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right) \tag{56}$$

2. For a single free nucleon (using Lorentz covariance, conservation of parity, time-reversal invariance, and isospin invariance) the matrix elements of the vector and axial-vector currents, in spin-isospin space, are written as

$$\begin{aligned} \langle \mathbf{k}'\lambda'; 1/2m_{t'} | J_\mu(0)_{TM_T} | \mathbf{k}\lambda; 1/2m_t \rangle \;&=\; i\bar{u}(\mathbf{k}'\lambda') \left[F_1^{(T)}\gamma_\mu + F_2^{(T)}\sigma_{\mu\nu}q_\nu + iF_S^{(T)}q_\mu \right] u(\mathbf{k}\lambda) \\ &\times\; \langle 1/2m_{t'} | I_T^{M_T} | 1/2m_t \rangle \end{aligned} \tag{57}$$

$$\begin{aligned} \langle \mathbf{k}'\lambda'; 1/2m_{t'} | J^5_\mu(0)_{TM_T} | \mathbf{k}\lambda; 1/2m_t \rangle \;&=\; i\bar{u}(\mathbf{k}'\lambda') \left[F_A^{(T)}\gamma_5\gamma_\mu - iF_P^{(T)}\gamma_5 q_\mu - F_T^{(T)}\gamma_5\sigma_{\mu\nu}q_\nu \right] u(\mathbf{k}\lambda) \\ &\times\; \langle 1/2m_{t'} | I_T^{M_T} | 1/2m_t \rangle \end{aligned} \tag{58}$$

The plane-wave single-nucleon states are labelled with 3-momenta $\mathbf{k}(\mathbf{k}')$, helicities $\lambda(\lambda')$ and isospins $1/2m_t(1/2m_{t'})$. The single-nucleon form factors $F_X^{(T)} = F_X^{(T)}(q_\mu^2)$, with $T = 0, 1$, and $X = 1, 2, S, A, P, T$ (vector (Dirac), vector (Pauli), scalar, axial, pseudoscalar,and tensor) are all functions of the momentum transfer q_μ^2.

Appendix B

The radial part of the wave functions in a (three-dimensional isotropic) harmonic oscillator potential is written as

$$R_{nl} \;=\; N_{nl} x^l e^{-x^2/2} \mathcal{L}_n^{l+1/2}(x^2), \tag{59}$$

where $x = r/b$, with b is the harmonic oscillator size parameter and N_{nl} the normalization factor

$$N_{nl}^2 \;=\; \frac{2n!}{b^3 \Gamma(n+l+3/2)}. \tag{60}$$

$\Gamma(x)$ denotes the known gamma function and $\mathcal{L}_n^{l+1/2}$ represent the Laguerre polynomials defined by:

$$\mathcal{L}_n^{l+1/2}(x) \;=\; \sum_{m=0}^{n} \Lambda_m(nl)\, x^m \;=\; \sum_{m=0}^{n} \frac{(-)^m}{m!} \binom{n+l+1/2}{n-m} x^m \,. \tag{61}$$

In Sect. 3, by writing $|n(l1/2)jm_j\rangle$, we mean $R_{nlj}(r)[Y_l(\Omega_r) \otimes \chi_{1/2}]^j_{m_j}$ with $\chi_{1/2}$ the Pauli spinor. We note that for the harmonic oscillator it holds $R_{nlj}(r) = R_{nl}(r)$. The adopted, in the present work, sequence of single-particle states is: 0s1/2, 0p3/2, 0p1/2, ...

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$n_1l_1 - n_2l_2$	L	$\mu=0$	$\mu=1$	$\mu=2$	$\mu=3$
$0d-0d$	0	1	-$\frac{4}{3}$	$\frac{4}{15}$	
	2	$\frac{14}{15}$	-$\frac{4}{15}$		
	4	$\frac{4}{15}$			
$0f-0f$	0	1	-2	$\frac{4}{5}$	-$\frac{8}{105}$
	2	$\frac{6}{5}$	$\frac{24}{35}$	$\frac{8}{105}$	
	4	$\frac{44}{105}$	-$\frac{8}{105}$		
	6	$\frac{8}{105}$			
$1s-1s$	0	1	-$\frac{4}{3}$	$\frac{2}{3}$	
$1p-1p$	0	1	-2	$\frac{22}{15}$	-$\frac{4}{15}$
	2	$\frac{6}{5}$	$\frac{16}{15}$	-$\frac{4}{15}$	
$0d-1s$	2	1	-$\frac{8}{3}\sqrt{\frac{1}{10}}$	$\frac{4}{3}\sqrt{\frac{1}{10}}$	
$0d-0f$	1	$\frac{1}{3}\sqrt{14}$	-$\frac{4}{15}\sqrt{14}$	$\frac{4}{105}\sqrt{14}$	
	3	$\frac{6}{35}\sqrt{14}$	-$\frac{4}{105}\sqrt{14}$		
$0d-1p$	1	-$\frac{2}{3}$	$\frac{6}{5}$	-$\frac{4}{15}$	
	3	-$\frac{8}{15}$	$\frac{4}{15}$		
$1s-1p$	1	$\frac{\sqrt{10}}{3}$	-2$\sqrt{\frac{2}{5}}$	$\frac{3}{2}\sqrt{\frac{2}{5}}$	
$0f-1p$	2	$\frac{4}{15}\sqrt{14}$	$\frac{26}{105}\sqrt{14}$	-$\frac{4}{105}\sqrt{14}$	
	4	-$\frac{4}{105}\sqrt{14}$	$\frac{4}{105}\sqrt{14}$		

Table 1: Coefficients ε_μ^L which determine the radial nuclear moments $\langle n_1 l_1 | j_L(qr) r^\kappa | n_2 l_2 \rangle$ (see Eqs. (39) and (46)) in the 1s-0d and 1p-0f model space.

$n_1l_1 - n_2l_2$	L	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
$0d - 0d$	1	$-\frac{7}{6}$	$\frac{14}{15}$	$-\frac{2}{15}$	
	3	$-\frac{3}{5}$	$\frac{2}{15}$		
$0f - 0f$	1	$-\frac{3}{2}$	$\frac{9}{5}$	$-\frac{18}{35}$	$\frac{4}{105}$
	3	$-\frac{33}{35}$	$\frac{44}{105}$	$-\frac{4}{105}$	
	5	$-\frac{26}{105}$	$\frac{4}{105}$		
$1p - 1p$	1	$-\frac{5}{6}$	$\frac{19}{15}$	$-\frac{13}{15}$	$\frac{2}{15}$
$0d - 0f$	2	$-\frac{3}{5}\sqrt{\frac{7}{2}}$	$\frac{6}{5}\sqrt{\frac{2}{7}}$	$-\frac{2}{15}\sqrt{\frac{2}{7}}$	
	4	$-\frac{11}{15}\sqrt{\frac{2}{7}}$	$\frac{2}{15}\sqrt{\frac{2}{7}}$		
$0d - 1p$	2	0	$-\frac{3}{5}$	$\frac{2}{15}$	
$0f - 0d$	2	$-\frac{3}{5}\sqrt{\frac{7}{2}}$	$\frac{6}{5}\sqrt{\frac{2}{7}}$	$-\frac{2}{15}\sqrt{\frac{2}{7}}$	
	4	$-\frac{11}{15}\sqrt{\frac{2}{7}}$	$\frac{2}{15}\sqrt{\frac{2}{7}}$		
$0f - 1p$	3	$\frac{3}{5}\sqrt{\frac{2}{7}}$	$-\frac{13}{15}\sqrt{\frac{2}{7}}$	$\frac{2}{15}\sqrt{\frac{2}{7}}$	
$1p - 0d$	2	$\frac{14}{15}$	$-\frac{13}{15}$	$\frac{2}{15}$	
$1p - 0f$	3	$\frac{9}{5}\sqrt{\frac{2}{7}}$	$-\frac{17}{15}\sqrt{\frac{2}{7}}$	$\frac{2}{15}\sqrt{\frac{2}{7}}$	

Table 2: Geometrical coefficients $\zeta_\mu^-(n_1l_1n_2l_2, L)$ which determine the radial integrals $\langle n_1l_1|\theta_l^-(\rho)|n_2l_2\rangle$ of Eq. (37). For details see the text.

$n_1l_1 - n_2l_2$	L	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
$0d - 0d$	1	$\frac{1}{2}$	$\frac{4}{15}$	-$\frac{2}{15}$	
	3	$\frac{1}{15}$	$\frac{2}{15}$		
$0f - 0f$	1	$\frac{5}{6}$	-$\frac{1}{15}$	-$\frac{26}{105}$	$\frac{4}{105}$
	3	$\frac{9}{35}$	$\frac{16}{105}$	-$\frac{4}{105}$	
	5	$\frac{2}{105}$	$\frac{4}{105}$		
$1p - 1p$	1	$\frac{1}{6}$	$\frac{7}{15}$	-$\frac{7}{15}$	$\frac{2}{15}$
$0d - 0f$	2	$\frac{7}{6}\sqrt{\frac{2}{7}}$	$\frac{4}{15}\sqrt{\frac{2}{7}}$	-$\frac{2}{15}\sqrt{\frac{2}{7}}$	
	4	$\frac{1}{5}\sqrt{\frac{2}{7}}$	$\frac{2}{15}\sqrt{\frac{2}{7}}$		
$0d - 1p$	2	-$\frac{2}{5}$	-$\frac{1}{5}$	$\frac{2}{15}$	
$0f - 0d$	2	$\frac{1}{15}\sqrt{\frac{7}{2}}$	$\frac{8}{15}\sqrt{\frac{2}{7}}$	-$\frac{2}{15}\sqrt{\frac{2}{7}}$	
	4	-$\frac{1}{15}\sqrt{\frac{2}{7}}$	$\frac{2}{15}\sqrt{\frac{2}{7}}$		
$0f - 1p$	3	-$\frac{1}{5}\sqrt{\frac{2}{7}}$	-$\frac{7}{15}\sqrt{\frac{2}{7}}$	$\frac{2}{15}\sqrt{\frac{2}{7}}$	
$1p - 0d$	2	$\frac{4}{15}$	-$\frac{1}{5}$	$\frac{2}{15}$	
$1p - 0f$	3	-$\frac{1}{15}\sqrt{\frac{2}{7}}$	-$\frac{1}{5}\sqrt{\frac{2}{7}}$	$\frac{2}{15}\sqrt{\frac{2}{7}}$	

Table 3: Geometrical coefficients $\zeta^+_\mu(n_1l_1n_2l_2, L)$ which determine the radial integral $\langle n_1l_1|\theta^+_l(\rho)|n_2l_2\rangle$ of Eq. (37). For details see the text.